Bipartite Ramanujan Graphs

Xiaoyu He

March 24, 2017

1 Overview

Expanders are a special family of regular graphs with useful properties for computer science. Three equivalent definitions of an expander graph are: every subset of vertices has many neighbors outside itself, the nontrivial eigenvalues of its adjacency matrix are small, and random walks in the graph converge quickly. As a result, the problem of constructing optimal expander graphs is of great interest.

The Ramanujan graphs are the “best possible” expanders in the sense that the largest nontrivial eigenvalues of their adjacency matrices are at most $2\sqrt{d-1}$, which matches the theoretical lower bound of Alon and Boppana (see [12]). Until recently, Ramanujan graphs were only known to exist in degrees of the form $p + 1$, where $p \equiv 1 \pmod{4}$. These graphs come from the explicit algebraic construction of Lubotsky, Philips, Sarnak [9] and, independently, Margulis [11]. Then, a series of papers by Marcus, Spielman, and Srivastava [10] managed to show nonconstructively that the existence of Ramanujan graphs of every degree using what they call the method of “interlacing families.” This key ingredient allows them to control the roots of the expected characteristic polynomial of (the adjacency matrix) of a certain random graph, and thus to show that for at least one instance of the graph the nontrivial eigenvalues are all small.

To quickly construct Ramanujan graphs using the interlacing families method, a polynomial time algorithm for computing the associated expected characteristic polynomials is needed. Cohen [4] was able to provide this step and completely describe a polynomial time algorithm for computing Ramanujan graphs of all degrees.

This paper will introduce the theory of Ramanujan graphs and sketch the main ideas in the above proofs. Section 2 will introduce the properties of expanders and show that Ramanujan graphs are indeed the optimal ones. Section 3 will follow the argument of Lubotsky, Philips, Sarnak [9] to show that certain Cayley graphs of certain finite algebraic groups are Ramanujan graphs. Finally, Section 4 will describe the work of Marcus, Spielman, and Srivastava [10] to prove nonconstructively the existence of Ramanujan graphs of all degrees using the method of interlacing families.

Many of these results are extremely technical and we will only cover high-level ideas where necessary.
2 Introduction to Ramanujan Graphs

In this section we prove some basic properties of graph expansion, mention how these definitions extend to the bipartite case, and show that Ramanujan graphs are the best possible spectral expanders.

2.1 Expanders

There are several closely related notions of expanders; we follow the exposition of Alon and Spencer [2]. Intuitively an expander is a “sparse quasirandom graph,” where the main implication of being pseudorandom is having no unexpected clustering.

Fix positive integers \( n > d \). Let \( G = (V, E) \) be an undirected \( d \)-regular graph on \( n \) vertices, with multiple edges and self-loops allowed. Define the neighborhood \( N(W) \) of a vertex subset \( W \subseteq V \) to be the set of all vertices in \( V \setminus W \) adjacent to at least one member of \( W \).

**Definition 1.** We call \( G \) a \( c \)-vertex expander if for every \( W \subset V \) with \( |W| \leq \frac{1}{2}|V| \), the inequality 
\[
|N(W)| \geq c|W|
\]
holds.

Intuitively, every small subset \( W \) expands by at least a constant factor \((1 + c)\) in size if we add its neighborhood. Obviously if \( W \) is almost all of \( V \) then it has little room to expand, so the condition \( |W| \leq \frac{1}{2}|V| \) is necessary for a useful definition.

It is equally reasonable to focus on edge expansion rather than vertex expansion. If \( B, C \) are disjoint subsets of \( V \), let \( e(B, C) \) count the number of edges between vertices of \( B \) and \( C \).

**Definition 2.** We call \( G \) a \( \gamma \)-edge expander if for every \( W \subset V \) with \( |W| \leq \frac{1}{2}|V| \), the inequality 
\[
|e(W, V \setminus W)| \geq \gamma d|W|
\]
holds.

That is, a constant fraction of the edges incident to \( W \) leave this same set. By degree considerations it is clear that a \( \gamma \)-edge expander is also a \( \gamma \)-vertex expander.

Let \( A_G \) be the adjacency matrix of \( G \), where \((A_G)_{u,v}\) is the number of edges between \( u \) and \( v \) in \( G \). Another reasonable notion for \( G \) being a “quasirandom graph” is for random walks on \( G \) to converge rapidly to the uniform distribution on \( G \); equivalently, the properly normalized powers of \( A_G \) to converge rapidly to the all-ones matrix. By the spectral theorem, this is equivalent to asking for \( A_G \) to have one eigenvalue much larger than the others in absolute value. Of course, a \( d \)-regular graph always has largest eigenvalue \( d \). This motivates our second notion of an expander.

**Definition 3.** We call \( G \) a \( \lambda \)-spectral expander if every eigenvalue of \( A_G \) except \( d \) has absolute value at most \( \lambda \).

If \( G \) is a \( \lambda \)-spectral expander then
\[
\|A_G^m - d^m J_n\|_\infty = O(\lambda^m),
\]
where \( \| \cdot \|_\infty \) is the sup norm on the entries of a matrix, and \( J_n \) is the \( n \times n \) all-ones matrix. Thus a random walk on \( G \) will converge exponentially rapidly to the uniform distribution.
The main goal of this section is to show that these three definitions of expanders are closely related. In fact we only prove some of the implications - the main result being the so-called Expander Mixing Lemma.

**Lemma 4. (Expander Mixing Lemma)** If $G = (V, E)$ is a $\lambda$-spectral expander on $n$ vertices, then for every partition of $V$ into two disjoint subsets $B, C$, the number of edges $e(B, C)$ between them satisfies

$$|e(B, C) - \frac{d|B||C|}{n}| \leq \lambda \sqrt{|B||C|}.$$

**Proof.** Let $A = A_G$ be the adjacency matrix of $G$, and let $e_v$ be the $v$-th standard basis vector of $\mathbb{R}^V$, where $v$ ranges through the vertices of $G$, and for a subset $S \subset V$ write

$$e_S = \sum_{v \in S} e_v.$$

Then the number of edges between $B$ and $C$ is given by the product

$$e(B, C) = e_B^T A e_C,$$

which is best understood by expanding $e_B$ and $e_C$ as linear combinations of a basis of eigenvectors for $A$. In particular the first eigenvector is just $e_V$ with eigenvalue $d$, so if we subtract the projection onto that component,

$$e_B^\perp = e_B - \frac{|B|}{n} e_V,$$

$$e_C^\perp = e_C - \frac{|C|}{n} e_V,$$

whence the product for $e(B, C)$ is

$$e(B, C) = (\frac{|B|}{n} e_V + e_B^\perp)^T A (\frac{|C|}{n} e_V + e_C^\perp)$$

$$= \frac{d|B||C|}{n^2} e_V^T e_V + (e_B^\perp)^T A e_C^\perp.$$

But $\|e_V\|^2 = n$, so

$$\left|e(B, C) - \frac{d|B||C|}{n}\right| \leq (e_B^\perp)^T A e_C^\perp$$

$$\leq \lambda \|e_B\| \|e_C\|$$

$$= \lambda \sqrt{|B||C|},$$

as desired. \(\square\)

**Remark.** Note that $\frac{d}{n} |B||C|$ is the expected number of edges in a random $d$-regular graph between two subsets $B$ and $C$, since the edge density is $\frac{d}{n}$.

**Corollary 5.** A $\lambda$-spectral expander is a $\gamma$-edge expander (and thus also a $\gamma$-vertex expander) for

$$\gamma = \frac{d - \lambda}{2d}.$$
Proof. By the Expander Mixing Lemma, if $|W| \leq \frac{1}{2}|V|$ and $G$ is a $\lambda$-spectral expander, then

$$e(W, V \setminus W) \geq \frac{d - \lambda}{n}|W|(n - |W|) \geq \frac{d - \lambda}{2d}d|W|.$$

Various partial converses have been given to this result, showing that a good vertex expander also has good spectral expansion. One such statement is a theorem of Alon [1], which we state without proof.

**Theorem 6.** A $c$-vertex expander is also a $\lambda$-spectral expander for

$$\lambda = d - \frac{c^2}{4 + 2c^2}.$$

Henceforth, we only deal with spectral expanders. By a $\lambda$-expander we will mean a $\lambda$-spectral expander. Sometimes we will also call $G$ an $(n, d, \lambda)$-expander if it has $n$ vertices, is $d$-regular, and is a $\lambda$-expander.

### 2.2 Bipartite Expanders

Now let $G$ be a $d$-regular bipartite graph on $n$ vertices. Then $G$ will never be a $\lambda$-expander in the sense of Definition 3, since $A_G$ will have two large eigenvalues $\pm d$. In fact, a random walk will never converge to the uniform distribution on $G$; instead, the odd numbered steps of the walk are supported in one part of $G$, and the even numbered steps on the other.

**Definition 7.** We say $G$ is a bipartite $\lambda$-expander if $G$ is bipartite and all of its eigenvalues other than $\pm d$ are at most $\lambda$ in absolute value.

It is easy to verify the following bipartite version of the Expander Mixing Lemma.

**Theorem 8.** (Expander Mixing Lemma, bipartite version.) If $G$ is a bipartite $\lambda$-expander with bipartition $L \cup R$, and $B \subseteq L$, $C \subseteq R$ are subsets of the two parts, then

$$\left|e(B, C) - \frac{d|B||C|}{n}\right| \leq \lambda \sqrt{|B||C|}.$$

It similarly follows that a bipartite $\lambda$-expander has good vertex and edge expansion.

In Sections 4 and ?? we will only be able to construct bipartite Ramanujan graphs.

### 2.3 Ramanujan Graphs

Given $n, d$, for which $\lambda$ can we find $(n, d, \lambda)$-expanders? It is easy to get a cheap estimate for the order of $\lambda$. 
Lemma 9. If $G$ is an $(n, d, \lambda)$-expander then
\[
\lambda \geq \sqrt{\frac{(n-2d)d}{n}}.
\]

Proof. Note that the sum of the squares of the eigenvalues of $A_G$ is equal to $\text{tr}(A_G^2)$. But $A_G^2$ has all $d$'s along the diagonal, and so
\[
\sum_i \lambda_i^2 = nd.
\]

The largest eigenvalue contributes $d^2$ to the sum, and $2d^2$ in the bipartite case, so if $\lambda$ is the second largest eigenvalue, $n\lambda^2 \geq (n-2d)d$, and the inequality follows. \qed

Thus, when $n$ becomes very large the best we can hope for is $\lambda = O(\sqrt{d})$. We would like to find precisely the right constant in front of $\sqrt{d}$.

Since expanders are supposed to be pseudo-random graphs, it is natural to check the spectrum of a random $d$-regular graph. In fact, it is a theorem of Friedman [6] that a random $d$-regular graph is a good expander.

We first need to define a random $d$-regular graph. Let $d$ be even, and pick independently and uniformly $d/2$ permutations $\pi_1, \ldots, \pi_{d/2}$ on the vertex set $V = \{1, \ldots, n\}$. Then define
\[
E = \{(i, \pi_j(i))|i = 1, \ldots, n; j = 1, \ldots, d/2\},
\]
yielding an undirected graph $G = (V, E)$ with possible multiple edges and self-loops. Counting self-loops twice, $G$ is $d$-regular.

Theorem 10. Let $\epsilon > 0$. With the above definition, a random $d$-regular graph on $n$ vertices is a $\lambda$-expander for
\[
\lambda = 2\sqrt{d-1} + \epsilon
\]
with probability $1 - O(n^{-\tau})$, where $\tau = \lceil(\sqrt{d-1}+1)/2\rceil - 1$, the implicit constant depending only on $\epsilon$.

It follows that there exist arbitrarily large $(2\sqrt{d-1} + \epsilon)$-expanders for any $\epsilon > 0$.

Conversely, Nilli showed that one cannot do better than $2\sqrt{d-1}$ as $n \to \infty$. The diameter of a graph $G$ is the largest distance between two vertices of $G$.

Theorem 11. [12] There exists a constant $c$ such that for any connected $d$-regular graph $G$ with diameter $\Delta$, $G$ is not a $\lambda$-expander for
\[
\lambda = 2\sqrt{d-1} \left(1 - \frac{c}{\Delta^2}\right).
\]

Note that the diameter of a $d$-regular graph must grow to infinity as $n \to \infty$, so we see that $\lambda = 2\sqrt{d-1}$ is optimal for spectral expanders.

Corollary 12. If $(G_m)_{m=1}^\infty$ is an infinite family of $d$-regular graphs with $|V(G_m)| \to \infty$ and every $G_m$ is a $\lambda$-expander, then $\lambda \geq 2\sqrt{d-1}$.

A $2\sqrt{d-1}$-expander is called a Ramanujan graph. The same results can be shown for bipartite expanders, and so we define a bipartite $2\sqrt{d-1}$-expander to be a bipartite Ramanujan graph.

We will dedicate the next sections to proving the existence of arbitrarily large bipartite Ramanujan graphs of every fixed degree $d$. 

3 Ramanujan Cayley Graphs

The first construction of an infinite family of Ramanujan graphs was given by Lubotsky, Philips, and Sarnak [9], and independently by Margulis [11]. Here we prove the following special case. Recall that the Legendre symbol \((\frac{q}{p})\) is defined by

\[
(\frac{q}{p}) = \begin{cases} 
1 & q \text{ is a square modulo } p \\
-1 & \text{otherwise.}
\end{cases}
\]

**Theorem 13.** Let \(p, q\) be two distinct primes that are \(1 \pmod{4}\), and suppose further that \((\frac{q}{p}) = -1\). Then there exists a bipartite Ramanujan graph \(X^{p,q}\) of degree \(d = p + 1\) and order \(n = q(q^2 - 1)\).

In particular, by Dirichlet’s theorem for any fixed \(p \equiv 1 \pmod{4}\) there are infinitely many \(q\) satisfying the conditions of Theorem 13, so there are arbitrarily large bipartite Ramanujan graphs of degree \(d = p + 1\). We will first collect some preliminary results about Cayley graphs, algebraic groups, and number theory.

### 3.1 Construction of \(X^{p,q}\)

A Cayley graph is a regular graph coming from a group and a set of generators.

**Definition 14.** Let \(G\) be a group and \(S \subset G\) be a set of generators for \(G\) such that \(a^{-1} \in S\) whenever \(a \in S\). Then the (undirected) Cayley graph \(\Gamma(G; S)\) for \(G\) with respect to \(S\) is the graph on vertex set \(G\) where \(g_1, g_2\) are connected by an edge iff \(g_1^{-1}g_2 \in S\).

Clearly a Cayley graph is always regular, since its degree is just \(|S|\), and connected, since \(S\) generates \(G\). We will pick \(G = PGL_2(\mathbb{F}_q)\), which is just the group of \(2 \times 2\) invertible matrices where matrices are identified with all their nonzero scalar multiples. We make a quick calculation to verify the size of \(G\).

**Lemma 15.** For any prime \(q\), \(|PGL_2(\mathbb{F}_q)| = q(q^2 - 1)|\).

**Proof.** The total number of \(2 \times 2\) invertible matrices over \(\mathbb{F}_q\) can be enumerated as follows. First pick a nonzero vector for the first row in \(q^2 - 1\) ways. Then, pick a linearly independent vector for the second row in \(q^2 - q\) ways. Finally, divide by \(q - 1\) scalar multiplies identified under projectivization. \(\square\)

The choice of \(S\) is much more involved and requires the following theorem of Jacobi, which strengthens Legendre’s Four-Square Theorem.

**Theorem 16.** (Jacobi) If \(n\) is a positive integer, let \(r_4(n)\) denote the number of representations \(n = a_0^2 + a_1^2 + a_2^2 + a_3^2\) as the sum of four squares. Then,

\[
r_4(n) = 8 \sum_{d|n, 4|d} d,
\]

and in particular \(r_4(p) = 8(p + 1)\) if \(p\) is a prime.
Thus there are $8(p+1)$ ways of writing $p = a_0^2 + a_1^2 + a_2^2 + a_3^2$. Since $p \equiv 1 \pmod{4}$, and squares are 0 or 1 (mod 4), it must be that one of the $a_i$ is odd and the rest are even. Thus, permuting the $a_i$ and switching the sign of $a_1$ if necessary, we get the following corollary.

**Corollary 17.** If $p$ is a prime $p \equiv 1 \pmod{4}$, then there are $p+1$ representations $p = a_0^2 + a_1^2 + a_2^2 + a_3^2$ with $a_0 > 0$ odd, and $a_1, a_2, a_3$ even.

Since $p \equiv 1 \pmod{4}$ we can pick some $i^2 \equiv -1 \pmod{p}$. To each solution $\alpha = (a_0, a_1, a_2, a_3)$ in Corollary 17, associate $\tilde{\alpha} \in PGL_2(\mathbb{F}_q)$ given by

$$\tilde{\alpha} = \begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{pmatrix}.$$ 

Let $S_p = \{ \tilde{\alpha} : \alpha$ is a solution in Corollary 17$\}$. Then the desired Ramanujan graph is $X_{p,q} = \Gamma(PGL_2(\mathbb{F}_q), S_p)$.

### 3.2 Sketch of Argument

We make a quick sketch of how Lubotsky, Philips, and Sarnak control the graph $X_{p,q}$ as defined above. The point is that the matrix

$$\begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{pmatrix}$$

can be mapped to the Hamiltonian quaternion $a_0 + ia_1 + ja_2 + ka_3$. This is hardly a surprise given that the proof of Jacobi’s theorem is best seen as an argument about quaternions. It remains to understand the multiplicative structure of the quaternions $\mathbb{H}$, in particular the multiplicative submonoid generated by all $p+1$ numbers which have quaternion norm $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$.

Next, the projectivization of matrices corresponds to identifying (real) multiples of quaternions, and the multiplicative monoid described above becomes a group $\Lambda(2)$ after identifying scalar multiples (which will always be powers of $p$ because of the norm). The generating elements come in pairs $\alpha, \alpha^{-1}$ conjugate, and therefore inverse to each other now that scalars are identified. Then, $\Lambda(2)$ is generated freely by these $(p+1)/2$ pairs, and its Cayley graph is just the infinite $(p+1)$-regular tree, which is the universal cover of all $(p+1)$-regular graphs.

Thus far all we have done is construct a group $\Lambda(2)$ and a set of generators $S_p$ such the corresponding Cayley graph $\Gamma(\Lambda(2), S_p)$ is the universal $(p+1)$-regular graph. It remains to pick the right subgroup to quotient by to get the right quotient of the corresponding Cayley graph. When we quotient by a prime $q \equiv 1 \pmod{4}$ everything works out, and when $q$ is a nonresidue mod $p$, we can furthermore guarantee that the resulting quotient is bipartite.

### 4 Interlacing Families

In this section we describe the construction of Marcus, Spielman and Srivastava [10] of Ramanujan graphs of all degrees $d \geq 3$. Whereas the previous argument relied on algebraic and number-theoretic considerations, this one is probabilistic in nature.
Again, it is easy to construct an individual small Ramanujan graph of a given degree $d$, for example $K_{d+1}$. When we say construct Ramanujan graph of a degree, we mean construct an infinite family of them with size going to infinity. The argument starts by rehashing an idea of Bilu and Linial [3], that of showing that certain 2-lifts of Ramanujan graphs are also Ramanujan. Once we show this is the case, we can generate an infinite family of Ramanujan graphs of degree $d$ by repeatedly lifting the complete graph $K_{d+1}$.

**Theorem 18.** Every (bipartite) Ramanujan graph with degree $d \geq 3$ has a Ramanujan 2-lift.

### 4.1 2-Lifts

Given a graph $G = (V, E)$, a 2-lift of $G$ is a graph which is a topological double cover of $G$.

**Definition 19.** For a graph $G = (V, E)$, a 2-lift of $G$ to be a graph which has two vertices $u_0, u_1$ (called the fibre of $u$) for each vertex $u$ of $G$, such that every edge $(u, v)$ in $G$ corresponds to two edges in the 2-lift. The 2-lift contains as edges either

$$\{(u_0, v), (u_1, v)\} \text{ or } \{(u_0, v), (u_1, v_0)\}.$$

Thus a 2-lift of a graph is a graph with twice as many vertices and twice as many edges. A 2-lift is not uniquely defined unless $G$ has no edges - there is a choice of how to lift every edge of $G$. Also note that the 2-lift is $d$-regular if $G$ is.

Bilu and Linial [3] showed that eigenvalues of a 2-lift of $G$ can be controlled by applying a “signing” to the edges of $G$: a function $s : E \rightarrow \{\pm 1\}$. In particular, given a 2-lift, an edge $(u, v)$ is signed $+1$ if the first of the two cases above is true, and $-1$ otherwise. Define the signed adjacency matrix $A_s$ to be the adjacency matrix with the sign $s(u, v)$ in each entry instead of always $+1$.

**Lemma 20.** The eigenvalues of the 2-lift of $G$ corresponding the signing $s$ are exactly the union (taken with multiplicity) of the eigenvalues of $A$ and the eigenvalues of $A_s$.

**Proof.** Let $B_s$ be the $2n \times 2n$ adjacency matrix of the two-lift of $G$ corresponding to a given signing. Then, break $B_s$ into an $n \times n$ array of $2 \times 2$ blocks, indexed by the fibres of pairs of vertices $(u, v)$ of $G$. Clearly the $(u, v)$-th block in $B_s$ is

$$
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{cases}
1 & s(u, v) = 1 \\
0 & s(u, v) = -1
\end{cases}.
$$

Thus, given any eigenvector $v = (v_1, \ldots, v_n)^T$ of $A$, we can find an eigenvector $v' = (v_1, v_1, v_2, v_2, \ldots, v_n, v_n)^T$ of $B_s$ by repeating each coordinate, and the eigenvalue is the same. This proves the first half of the Lemma.

Next, given any eigenvector $v = (v_1, \ldots, v_n)^T$ of $A_s$, define $v' = (v_1, -v_1, v_2, -v_2, \ldots, v_n, -v_n)^T$. This is an eigenvalue of $B_s$, and each $2 \times 2$ block either fixes the corresponding two coordinates $(v_i, -v_i)$ or negates it by flipping the coordinates $(-v_i, v_i)$, exactly depending on $s(u, v)$. It follows that $v'$ also has the same eigenvalue for $B_s$ as $v$ does for $A_s$.

In general we have to argue about generalized eigenvectors to account for possible multiplicity but it’s not difficult.
We call the eigenvalues of $A$ the old eigenvalues and the eigenvalues of $A_s$ the new eigenvalues. With this result in hand, it remains to show that there exists a signing $s$ for which $A_s$ has no eigenvalues $> 2\sqrt{d-1}$ to find a Ramanujan 2-lift of a Ramanujan graph.

### 4.2 The Matching Polynomial

The main object to control is called the matching polynomial of a graph.

**Definition 21.** Let $G = (V, E)$ be a graph on $n$ vertices, and let $m_i$ be the number of matchings on $G$ with $i$ edges ($m_0 = 1$). Then, the matching polynomial of $G$ is defined as

$$\mu_G(x) = \sum_{i \geq 0} (-1)^i m_i x^{n-2i}.$$ 

In particular $\deg(\mu_G) = n$. The matching polynomial has the following useful properties, due to Heilman and Lieb [8].

**Theorem 22.** (Theorems 4.2 and 4.3, [8]) For every graph $G$, $\mu_G(x)$ has only real roots, and if the maximum degree of $G$ is $d$, then all the roots of $\mu_G(x)$ have absolute value at most $2\sqrt{d-1}$.

The matching polynomial is related to 2-lifts and signings by the following remarkable theorem of Godsil and Gutman [7]. A random signing is a signing $s : E \rightarrow \{\pm 1\}$ where every edge is signed $\pm 1$ with equal probability independently.

**Theorem 23.** Let $G = (V, E)$ be a graph, $s : E \rightarrow \{\pm 1\}$ be a random signing, and let $f_s$ be the characteristic polynomial of $A_s$. Then,

$$\mathbb{E}_s[f_s(x)] = \mu_G(x).$$

**Proof.** Let $S_n$ be the set of permutations on $1, \ldots, n$, and $S(T)$ the set of permutations on $T$. Then, by simply expanding the determinant formula and applying linearity of expectation,

$$\mathbb{E}_s[f_s(x)] = \mathbb{E}_s[\det(xI - A_s)]$$

$$= \mathbb{E}_s \left[ \sum_{\sigma \in S_n} (-1)^{|\sigma|} \prod_{i=1}^{n} (xI - A_s)_{i, \sigma(i)} \right]$$

$$= \sum_{k=0}^{n} x^{n-k} \sum_{T \subset [n], |T|=k} \sum_{\pi \in S(T)} \mathbb{E}_s \left[ (-1)^{|\pi|} \prod_{i \in T} (A_s)_{i, \pi(i)} \right]$$

But $(A_s)_{i, \pi(i)}$ is either always 0 if $i, \pi(i)$ is not an edge, or else a random $\pm 1$ independent of all the other entries of $A_s$ except $\pi(i), i$. Thus the expectancies inside all vanish except those in which $\pi(\pi(i)) = i$ for all $i$ but $\pi(i) \neq i$. Thus the inner expectancy is counting exactly the number of perfect matchings on $T$, the sign of $\pi$ is always $k/2$, and so the coefficient of $x^{n-k}$ is $(-1)^{k/2}m_{k/2}$, as desired. \[\Box\]
4.3 Interlacing Families

So far, we have shown that the expected characteristic polynomial of $A_s$ is exactly the matching polynomial $\mu_G$, and this latter polynomial has all roots bounded by $2\sqrt{d-1}$. It remains to show that among all possible $f_s$, at least one has largest root bounded by the largest root of $\mathbb{E}[f_s]$. We prove in particular that this holds in general for interlacing families of polynomials.

**Definition 24.** We say a polynomial $g = c\prod_{i=1}^{n-1}(x - \alpha_i)$ interlaces with a polynomial $f = c'\prod_{i=1}^{n}(x - \beta_i)$ if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n.$$  

We say $g$ is a common interlacing for $f_1, \ldots, f_k$ if it interlaces all of them.

The classic example of interlacing is the derivative $f'$ always interlacing with a polynomial $f$ if it has all distinct real roots. Now the main result on interlacings gives us exactly what we want.

**Lemma 25.** (Lemma 4.2, [10]) Let $f_1, \ldots, f_k$ be real-rooted polynomials of the same degree and positive leading coefficients. Define

$$f_\emptyset = \sum_{i=1}^{k} f_i.$$  

If $f_1, \ldots, f_k$ have a common interlacing, then there exists $f_i$ whose largest real root is at most the largest real root of $f_\emptyset$.

**Proof.** This follows from the intermediate value theorem. Each $f_i$ interlaces with a one-lower degree polynomial $g$, and so the largest root of $f_i$ is larger than the largest root $\alpha$ of $g$. Since the leading coefficients are all positive, and $f_i$ changes sign exactly once past $\alpha$, this means that $f_i(\alpha) < 0$. But then $f_\emptyset(\alpha) < 0$ and $f_\emptyset$ has at least one root $\beta > \alpha$.

But then $\sum f_i(\beta) = 0$, so there is some $f_i$ with $f_i(\beta) \geq 0$. Since $f_i$ has exactly one root past $\alpha$, this root must be before $\beta$, as desired. \qed

We can apply this Lemma repeatedly to a family of polynomials where every step there is an common interlacing.

**Definition 26.** Let $S_1, \ldots, S_m$ be finite sets and for every assignment $s_1, \ldots, s_m \in S_1 \times \cdots \times S_m$ let $f_{s_1,\ldots,s_m}(x)$ be a real-rooted degree $n$ polynomial with positive leading coefficient. For a partial assignment $s_1, \ldots, s_k \in S_1 \times \cdots \times S_k$, define

$$f_{s_1,\ldots,s_k} = \sum_{s_{k+1},\ldots,s_m} f_{s_1,\ldots,s_m},$$

and $f_\emptyset$ corresponds to the empty partial assignment. We say $\{ f_{s_1,\ldots,s_m} \}$ form an interlacing family if for all $k = 0, \ldots, m-1$, and all $s_1, \ldots, s_k \in S_1 \times \cdots \times S_k$, the polynomials

$$\{ f_{s_1,\ldots,s_k,t} \}_{t \in S_{k+1}}$$

have a common interlacing.
It follows by applying Lemma 25 \( m \) times that every interlacing family has the same property.

**Theorem 27.** Let \( \{ f_{s_1}, \ldots, s_m \} \) be an interlacing family as in Definition 26. Then, there is some \( s_1, \ldots, s_m \in S_1 \times \cdots \times S_m \) for which the largest root of \( f_{s_1}, \ldots, s_m \) is at most that of \( f_\emptyset \).

There is a simple criterion for interlacing in the literature, see for example Dedieu [5].

**Lemma 28.** Let \( \{ f_s \} \) be polynomials of the same degree with positive leading coefficients. Then \( f_1, \ldots, f_k \) have a common interlacing iff every convex combination \( \sum \lambda_i f_i \) is real-rooted.

### 4.4 Signing Polynomials are Interlacing

It remains to show that the family of characteristic polynomials of signed adjacency matrices for a given graph \( G \) \( \{ f_s \}_{s \in \{0,1\}^E} \) form an interlacing family. Using Lemma 28, it suffices to show the following generalization of Theorem 23.

**Lemma 29.** Let \( p_1, \ldots, p_s \) be numbers in \([0,1]\). Then, the following polynomial is real-rooted:

\[
\sum_{s \in \{\pm1\}^m} \left( \prod_{i: s_i = 1} p_i \right) \left( \prod_{i: s_i = -1} (1 - p_i) \right) f_s(x).
\]

This is a moderately technical determinant calculation that we will skip.

### 4.5 Summary

We proved that every bipartite Ramanujan graph with degree \( d \geq 3 \) has a Ramanujan 2-lift; in fact, we proved the stronger assertion that every bipartite graph \( G \) has a 2-lift such that all of the new eigenvalues are \( \leq 2\sqrt{d - 1} \).

Note that we only bounded the nontrivial eigenvalues above, and crucially needed the fact that in a bipartite graph, the eigenvalues are symmetric about 0 to get \( \lambda_2 \geq -2\sqrt{d - 1} \). Thus this argument fails to generalize to the non-bipartite setting.

The first step was to show that new eigenvalues introduced by the 2-lift are just the eigenvalues of the corresponding signed adjacency matrix \( A_s \).

Then, we controlled these eigenvalues by looking at the expected characteristic polynomial and noted that this expected characteristic polynomial is just the matching polynomial \( \mu_G(x) \), which has the desired property of all roots being \( \leq 2\sqrt{d - 1} \).

Of course, in general the roots of the average of a family of polynomials has very little to do with the roots of any one of them. However, we showed that in the special case where the family has common interlacings, at least one of them has roots bounded by those of the average.

Finally, we proved that the family of signed characteristic polynomials indeed satisfy this interlacing families property, so one of them is as nice as \( \mu_G \).

In principle, this gives an algorithm for determining the Ramanujan 2-lift: at each interlacing step, check each of the partially determined expected characteristic polynomials \( f_{s_1}, \ldots, s_{k+1} \) for one with no large roots, where \( t \in \{\pm1\} \). Set \( s_{k+1} \) to this value of \( t \) and continue. The work of Cohen [4] gives a polynomial time algorithm for computing exactly these partially determined expected characteristic polynomials of \( A_s \), which makes this construction polynomial time computable.
References


