1 Introduction

The purpose of this article is to revisit the triangle problem of Heilbronn, which asks:

Problem 1. Let $K$ be a convex region in the plane, and $\triangle(K, n)$ be the maximal minimum area of some triangle among $n$ points in $K$. What is the asymptotic growth rate of $\triangle(K, n)$?

It is easy to see that the exact shape of $K$ is irrelevant (up to constants), so we will suppress the dependence on $K$ and just write $\triangle(n)$.

Theorem 2. The triangle function satisfies $n^{-2} \ll \triangle(n) \ll n^{-1}$.

Proof. If $n$ is prime, pick $n$ points in the “modular parabola” $(t/n, t^2/n) \pmod{1}$ in the unit square $K$. No three of these points lie on a line, so by Pick’s theorem any triangle among them has area at least $n^{-2}/2$. Alternatively pick the points randomly.

Subdivide the square $K$ into $n/3$ convex regions of area $3/n$; some three points lie in one of these regions, so their area is $1/n$. Alternatively, triangulate. \qed

Roth was the first to make serious progress past this point. We will first sketch the log log saving he made:

Theorem 3. The triangle function satisfies $\triangle(n) \ll n^{-1}(\log \log n)^{-1/2}$.

After this, Roth was able to find a surprising application of Fourier analytic techniques developed by analytic number theorists to get a power-saving upper bound.

Theorem 4. The triangle function satisfies $\triangle(n) \ll n^{-c}$ for some positive constant $c > 1.1$.

The exact constant in this argument was later optimized to $8/7$ by Komlos, Pintz, Szemeredi. They also showed the lower bound:

Theorem 5. The triangle function satisfies $\triangle(n) \gg n^{-2}\log(n)$.

This disproves the original conjecture of Heilbronn that the correct order of magnitude is $n^{-2}$. We will focus on the ideas of Roth.
2 Deficient Strips

Roth’s first idea is the following. Given any pair of points \( \tau = (p_1, p_2) \), if there are no triangles of small area then this forces a strip around \( \tau \), which he calls \( H_\tau(w) \), \( w \) being the width, to be empty of all other points from our set. Write \( d(\tau) \) to be the distance between the two points in \( \tau \) and \( \theta(\tau) \) to be the angle of the slope of \( \tau \). Write \( A = n^{-1}(\log \log n)^{-1/2} \).

**Fact 6.** If \( S \) is a set of \( n \) points with no triangles with area less than \( A \), then

\[
H_\tau(2A/d(\tau))
\]

contains no other points of \( S \) besides \( \tau \), for any pair \( \tau \) of points in \( S \).

The point is that these strips are really quite large, and there are an unexpectedly small number of points of \( S \) in each of them when \( d(\tau) \) is small. The problem is that these strips can intersect a lot. The second idea to combat this is to restrict to a family of pairs \( \tau \) where (a) every pair is close together \((n^{-1/2})\) and (b) the slopes of these pairs are all similar. Notice that if two strips are roughly parallel and they intersect in the region, then actually all of the triangles among these four points have relatively small area.

**Proof.** (of Theorem 3). There are at least \( \gg n \) pairs of points \( \tau \) such that \( d(\tau) \ll Cn^{-1/2} \). Also, we can pick at least \( \gg An^{3/2} \) of these so that

\[
\theta \in [\theta_0, \theta_0 + Cn^{1/2}A].
\]

Then each of these pairs generates a deficient strip \( H_\tau(Cn^{1/2}A) \) and no two of them can intersect. Each strip cuts out an area roughly proportional to its width, so the remaining area is \((1 - (\log \log n)^{-1})\) times the original area, which can be decomposed into \( \ll An^{3/2} \) convex regions, among which there are at least \( n - An^{3/2} = n(1 - n^{-1/2}(\log \log n)^{-1/2}) \) points left. Some such region will have at least the expected ratio of points to area, and we proceed by descent.

\[\square\]

3 A More Refined Approach

It is hard to do better immediately with only Fact 6 in hand. The problem is that the widths of these strips are much too small, and there is no way to show that such small strips should contain more than 2 points. Using \( d(\tau) \ll n^{-1/3} \), say, Fact 6 shows that all such pairs \( \tau \) have some strip \( H_\tau(w) \) very deficient in points, where \( w = A/d(\tau) \approx n^{-2/3} \).

On the other hand, what we can show is (qualitatively):

**Fact 7.** A significant number of pairs \( \tau \) for which \( d(\tau) \ll n^{-1/3} \) are such that \( H_\tau(w') \), where \( w' \) is appropriately large \((\approx n^{-1/4})\), contain the expected number of points:

\[
\frac{1}{[K]} \int H_\tau(w'; X)K(X)dX.
\]
The question is how to go from very thin strips $H_\tau(w)$ being very deficient in points of $S$, to much thicker strips $H_\tau(w')$ being somewhat deficient in points of $S$. We will only need to show that $H_\tau(w')$ is deficient of points on average.

The idea is as follows: we consider the function

$$\phi_\tau(X) = \frac{1}{w} H_\tau(w; X) - \frac{1}{w'} H_\tau(w'; X).$$